

Scaling of Ghosts and Quantitative Bifurcations

Christian Kuehn

Abstract

The scaling of the time delay near a “bottleneck” of a generic saddle-node bifurcation is well-known to be given by an inverse square-root law. We extend the analysis to several non-generic cases for smooth vector fields. We proceed to investigate C^0 vector fields and demonstrate that parameter nonlinearities and phase-space nonlinearities have distinct effects. Our main result is a new phenomenon, which we call “quantitative bifurcation” in two-parameter families having a saddle-node bifurcation upon changing the first parameter. We find distinct scalings for different values of the second parameter ranging from power laws with exponents in $(0, 1)$ to scalings given by $O(1)$. We illustrate the quantitative bifurcation by an overdamped pendulum with varying length.

1 Introduction

Saddle-node bifurcations have been extensively studied in dynamical systems. The normal form in the context of ordinary differential equations is

$$\dot{x} = r + x^2 \quad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{R} \quad (1)$$

where r is the bifurcation parameter. Solving for the fixed points we set $r + x^2 = 0$, which has two solutions $x_{\pm} = \pm\sqrt{-r}$ giving no fixed points for $r > 0$ and a non-hyperbolic fixed point $x_{\pm} = 0$ at $r = 0$. For $r < 0$ we obtain an attracting fixed point x_- and a repelling fixed point at x_+ , hence a saddle-node bifurcation occurs at $r = 0$. If we set $x(0) = x_0$ we can solve (1) by separation of variables and obtain

$$\int_{x_0}^{x(t)} \frac{1}{r + s^2} ds = \int_0^t dt \quad (2)$$

Using trigonometric substitution we get

$$\tan^{-1} \left(\frac{x(t)}{\sqrt{r}} \right) - \tan^{-1} \left(\frac{x_0}{\sqrt{r}} \right) = t\sqrt{r} \quad (3)$$

Therefore the time a trajectory spends in the interval $[-1, 1]$ is

$$\frac{2 \tan^{-1} \left(\frac{1}{\sqrt{r}} \right)}{\sqrt{r}} = t \quad (4)$$

Taking $r \rightarrow 0^+$ gives

$$t \sim \frac{1}{\sqrt{r}} \quad (5)$$

In the literature this scaling law is referred to as *intermittency*, “*bottleneck*” or *saddle-node ghost*. One of the earliest references is [7] where saddle-node bifurcations of maps are investigated. Textbook

references include [4] and [9]. Applications of the scaling law to physical systems can, for example, be found in [1], [3] and [8]. We remark that the square-root scaling is found for any generic saddle-node bifurcation in sufficiently smooth vector fields independent of the dimension of the phase space. To justify this statement, recall the following theorem (see [4]):

Theorem 1.1. *Consider $\dot{y} = g(\mu, y)$ with $\mu \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Assume that for $\mu = \mu_0$ there exists an equilibrium y_0 such that:*

1. $D_y g(\mu_0, y_0)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w .
2. $D_y g(\mu_0, y_0)$ has k eigenvalues of negative real part and $n - k - 1$ eigenvalues with positive real part.
3. $w((\partial g / \partial \mu)(y_0, \mu_0)) \neq 0$
4. $w(D_y^2 g(\mu_0, y_0)(v, v)) \neq 0$

then there is a smooth curve of equilibria in $\mathbb{R} \times \mathbb{R}^n$ passing through (μ_0, y_0) tangent to the hyperplane $\{\mu_0\} \times \mathbb{R}^n$ with no equilibrium on one side of the hyperplane for each value μ and 2 equilibria on the other side of the hyperplane for each μ . The two equilibria are hyperbolic and have stable manifolds of dimensions k and $k + 1$ respectively.

Furthermore the set of equations satisfying Theorem 1.1 is known to be open and dense in C^∞ one-parameter families with an equilibrium having a simple zero eigenvalue.

If we assume in addition that $D_y g(\mu_0, y_0)$ has no eigenvalues with positive real part we obtain that the dynamics near (μ_0, y_0) has $n - 1$ exponentially attracting directions outside of a 1-dimensional center manifold M . On M the dynamics is given up to a normal form transformation by $\dot{x} = \mu + x^2$ for $x \in M$ (see e.g. [5]).

Hence we can restrict to the equation

$$\dot{x} = f(r, x) \quad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{R}$$

We shall from now on consider one-dimensional flows only. Necessary conditions for a saddle-node at $(r, x) = (0, 0)$ are $f(0, 0) = 0 = f_x(0, 0)$. The generic sufficient conditions are given by:

$$f_{xx}(0, 0) \neq 0 \tag{6}$$

$$f_r(0, 0) \neq 0 \tag{7}$$

We can easily see, using the implicit function theorem for f_x at $(0, 0)$, that $f(x, 0)$ is locally quadratic. Furthermore $f_r(0, r)$ is locally linear, therefore the square-root scaling law is not affected by a change of coordinates transforming the equation $\dot{x} = f(r, x)$ into the normal form (1). Hence all smooth vector fields with generic saddle-node bifurcations scale in the same way with respect to a bottleneck occurring for $r \rightarrow 0^+$.

In the next section we briefly address the question, what happens to the scaling law when the saddle-node is non-generic. In this case either (6) or (7) are violated and we refer to such a situation as a degenerate saddle-node.

2 Degenerate Saddle-Nodes

We treat each of the cases for degenerate saddle-nodes in turn.

2.1 Case 1 - $f_r(0,0) = 0$

The linear crossing condition for r at $r = 0$ fails and we consider the equation

$$\dot{x} = R(r) + x^2 \quad \text{for } x \in \mathbb{R}, r \in \mathbb{R} \text{ and } R \in C^\infty(\mathbb{R})$$

without restrictions on the first derivative of R . In particular we can show:

Proposition 2.1. *Let $a : (0, \infty) \rightarrow \mathbb{R}^+$ with $a(r) = 1/r^k$ for $k \in \{2, 3, \dots\}$. Then there exists a function $R(r)$, $R \in C^\infty(\mathbb{R})$, such that the equation $\dot{x} = R(r) + x^2$ has a saddle-node bifurcation at $r = 0$. The scaling law for this saddle-node is given by $t \sim a(r)$ as $r \rightarrow 0^+$.*

Proof. Define $R(r) = 1/a(r)^2$ where we use the even extension for $1/a(r)^2$ to define $R(r)$ on the entire real line so that $R(r) = r^{2k}$. Note that $R(0) = 0$ and $R \in C^\infty(\mathbb{R})$. Then fix any $r_0 \in \mathbb{R}$ and set $R(r_0) = k > 0$. Using the same separation of variables argument as in (2)-(5) we consider

$$\dot{x} = k + x^2$$

and obtain the time a trajectory spends in $[-1, 1]$ as

$$t = \frac{2 \tan^{-1} \left(\frac{1}{\sqrt{k}} \right)}{\sqrt{k}} = 2a(r_0) \tan^{-1}(a(r_0))$$

so that since r_0 was arbitrary we get that $t \sim a(r) = r^k$ as $r \rightarrow 0^+$. \square

Remark: Note that we have included topologically degenerate cases such as $\dot{x} = r^2 + x^2$ in Proposition 2.1. Disregarding these cases excludes all scaling laws with even powers $k = 2m$.

2.2 Case 2 - $f_{xx}(0,0) = 0$

Consider the equation:

$$\dot{x} = r + F(x) \quad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{R}$$

We assume for now that $F \in C^\infty(\mathbb{R})$. Notice that we still require that $F(0) = 0 = F_x(0)$ as necessary conditions for a saddle-node at $(r, x) = (0, 0)$. Also we assume without loss of generality that $F(x) > 0$ for $x \neq 0$. Considering the Taylor expansion of F at 0 we get

$$F(x) = c_4 x^4 + c_6 x^6 + \dots + c_{2m} x^{2m} + O(x^{2m+1})$$

as $F_{xx}(0) = 0$. We assume that F has a non-zero term in its Taylor expansion and omit the case when F is C^∞ and has vanishing Taylor expansion at 0, i.e. is “completely flat”. To find the scaling law for the time a trajectory spends in $[-1, 1]$ we have to evaluate the integral:

$$\int_{-1}^1 \frac{1}{r + c_4 x^4 + c_6 x^6 + \dots + c_{2m} x^{2m} + O(x^{2m+1})} dx \sim \int_{-1}^1 \frac{1}{r + c_{2k} x^{2k}} dx \quad \text{for } r \rightarrow 0^+$$

where $2k$ is the smallest index with nonzero Taylor coefficient. Since the asymptotic behaviour of the time spend in $[-1, 1]$ is independent of the interval of non-zero length centered at 0 we can find the scaling law by solving the integration problem:

$$\int_{-\infty}^{\infty} \frac{1}{r + x^{2k}} dx \quad \text{for } k \in \{2, 3, 4, \dots\}$$

To evaluate the last integral we either use substitution twice or use contour integration over the contour in \mathbb{C} given by the semi-circle with radius a and the interval $[-a, a]$. In any case, we get

$$\int_{-\infty}^{\infty} \frac{1}{r + x^{2k}} dx \sim r^{-\frac{2k-1}{2k}} \quad \text{as } r \rightarrow 0^+$$

The discussion can be summarized in the following result:

Proposition 2.2. *Consider the ODE*

$$\dot{x} = r + F(x) \quad \text{with } F(0) = 0 = F_x(0, 0) = F_{xx}(0) \text{ and } F(x) > 0 \text{ for } x \neq 0 \quad (8)$$

Assume that F is real analytic. Define m to be the smallest exponent with nonzero coefficient in the Taylor expansion for F at 0, then (8) has a saddle-node bifurcation at $(r, x) = (0, 0)$ with scaling law given by

$$t \sim r^{-\frac{m-1}{m}}$$

2.3 Further Remarks

Although degenerate saddle-nodes are not “generic” in the space of C^∞ 1-parameter vector fields $\dot{x} = f(r, x)$ with $f(0, 0) = 0 = f_x(0, 0)$ they still might be observed in practical applications, e.g. due to symmetry inherent in the system or given nonlinear parameter dependencies.

Propositions 2.1 and 2.2 show that we should expect various power laws as scalings near a saddle-node bifurcation for a smooth vector field. The natural question arises what happens if we drop the smoothness requirements of our vector field. In particular we consider the case when the vector field is only continuous at $(r, x) = (0, 0)$.

3 Non-Smooth Saddle-Nodes

We define a saddle-node bifurcation for C^0 vector fields as a topological change of the phase portrait homeomorphic to the case of a smooth saddle-node bifurcation. We remark that the analytical description presented in Theorem 1.1 no longer applies and refer to [2] and [6] (and references therein) for analytic methods in the theory of non-smooth bifurcations. Suppose we drop the smoothness requirement on the parameter dependence and allow $R(r) \in C^0(\mathbb{R})$, then we can show:

Proposition 3.1. *Let $a : (0, \infty) \rightarrow \mathbb{R}^+$, $a(r) \in C^\infty((0, \infty))$ and $a(r) \rightarrow \infty$ as $r \rightarrow 0^+$. Then there exists a function $R(r)$, $R \in C^0(\mathbb{R})$, such that the equation $\dot{x} = R(r) + x^2$ has a saddle-node bifurcation at $r = 0$ with no equilibria for $r > 0$ and two equilibria for $r < 0$. The scaling law for this saddle-node is given by $t \sim a(r)$ as $r \rightarrow 0^+$.*

Proof. The proof is completely analogous to the proof of Proposition 2.1 with the additional observation that $R(r) = 1/a(r)^2 \in C^0(\mathbb{R})$ in the current setting. \square

Proposition 3.1 implies that any scaling law can occur if the parameter dependence on r is non-linear and only C^0 . This is in contrast to the fact that in the smooth case we expect power laws. For the rest of this paper we shall focus on the equation:

$$\dot{x} = r + F(x) \quad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{R}$$

with $F \in C^\infty(\mathbb{R} - \{0\})$ and $F \in C^0(\mathbb{R})$. We shall show that this case precisely gives an interesting “intermediate” behaviour between the cases considered so far. We introduce the notation:

$$\dot{x} = \begin{cases} r + f_-(x) & \text{for } x < 0 \\ r + f_+(x) & \text{for } x \geq 0 \end{cases} \quad (9)$$

with $f_\pm(0) = 0$, $f_\pm(x) > 0$ for $x \neq 0$, $f_- \in C^\infty((-\infty, 0))$ and $f_+ \in C^\infty((0, \infty))$. Observe that (9) has a saddle-node bifurcation at $(r, x) = (0, 0)$. If we use the same techniques to investigate the scaling law for $r \rightarrow 0^+$ as in (2)-(5) we obtain two integrals:

$$A_-(r) = \int_{-1}^0 \frac{1}{r + f_-(x)} dx \quad \text{and} \quad A_+(r) = \int_0^1 \frac{1}{r + f_+(x)} dx$$

Observe that if $f_-(x) \ll f_+(x)$ for $x \rightarrow 0$ then $A_-(r) \ll A_+(r)$ for $r \rightarrow 0^+$. So we can assume that $f_-(x) \sim f_+(x)$ as $x \rightarrow 0$. In particular we assume without loss of generality (with respect to investigating the scaling law) that $f_+(x)$ is given and $f_-(x)$ is its even extension to $(-\infty, 0)$. For simplicity of notation we shall simply denote $f_+(x) = f(x)$ for $x \geq 0$ and $f_-(x) = f(x)$ for $x < 0$, drop the subscripts \pm .

3.1 Three Examples

We compare three key examples of ODEs $\dot{x} = r + f_i(x)$ ($i = 1, 2, 3$) leading to further investigation. They are given by

$$f_1(x) = \sqrt{x} \quad f_2(x) = x \quad f_3(x) = x^2 \quad \text{for } x \geq 0 \quad (10)$$

where we use even extensions to define the functions f_i on \mathbb{R} . The examples are illustrated in Figure 1.

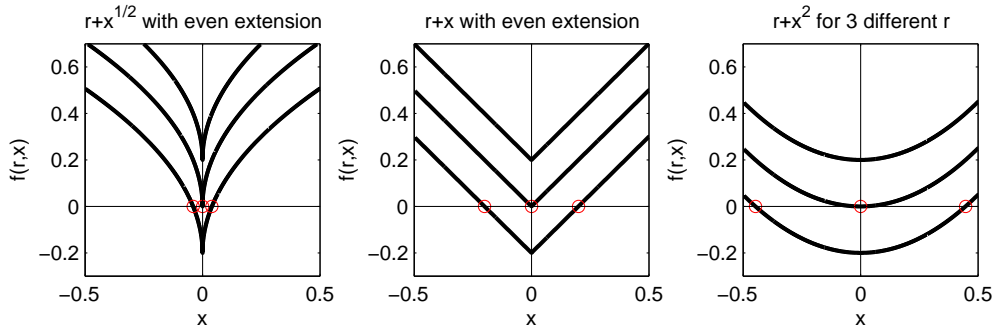


Figure 1: $f(x, r) = r + x^a$ for $r = -\frac{1}{2}, 0, \frac{1}{2}$ and $a = \frac{1}{2}, 1, 2$

Integrals giving the scaling laws are:

$$A_i(r) = \int_0^1 \frac{1}{r + f_i(x)} dx \quad \text{for } i = 1, 2, 3$$

The integrals can be evaluated explicitly and the results are:

$$A_1(r) = 2 + 2r \ln(r) - 2r \ln(1 + r) \quad A_2(r) = \ln\left(1 + \frac{1}{r}\right) \quad A_3(r) = \frac{\tan^{-1}\left(\frac{1}{\sqrt{r}}\right)}{\sqrt{r}}$$

Table 1: Scaling laws for three examples

Function	Scaling Law	Order of the Scaling
$f_1(x) = \sqrt{x}$	$t = 2$	constant
$f_2(x) = x$	$t \sim \ln(r)$	logarithmic
$f_3(x) = x^2$	$t \sim \frac{1}{\sqrt{r}}$	square-root

We consider the limit $r \rightarrow 0^+$ to get the scaling laws. The results are summarized in Table 1.

Therefore we should pose the question, what exponents α for the family $f_\alpha(r, x) = r + x^\alpha$ have scaling laws $O(1)$.

3.2 “Quantitative Bifurcation”

Theorem 3.2. *For each r , define $f_\alpha(r, x)$ as the even extension of*

$$x \mapsto r + x^\alpha \quad x \geq 0$$

Consider the two-parameter family of differential equations given by

$$\dot{x} = f_\alpha(r, x) \quad \text{for } x \in \mathbb{R} \text{ and } r \in \mathbb{R}$$

then we find the scaling law for $\alpha \in (0, 1)$ to be given by $t = O(1)$, whereas $t \rightarrow \infty$ for $\alpha \in [1, \infty)$, as $r \rightarrow 0^+$.

Proof. Let $r_n > 0$ be a sequence such that $r_n > r_{n+1}$ for all n and $r_n \rightarrow 0$ as $n \rightarrow \infty$. Then consider

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{1}{r_n + x^\alpha} dx \tag{11}$$

Notice that $g_n(x) = \frac{1}{r_n + x^\alpha}$ is a monotonically increasing sequence of functions on $[0, 1]$. Therefore we can apply the monotone convergence theorem in equation (11) to obtain:

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{1}{r_n + x^\alpha} dx = \int_0^1 \lim_{n \rightarrow \infty} \frac{1}{r_n + x^\alpha} dx = \int_0^1 \frac{1}{x^\alpha} dx$$

where the last integral converges for $\alpha \in (0, 1)$ and diverges for $\alpha \geq 1$. □

The main observation of Theorem 3.2 is that there is a major quantitative change in the behaviour of the solutions at $\alpha = 1$. In analogy with the classical terminology we refer to $\alpha = 1$ as a “quantitative bifurcation”. Notice that $\alpha = 1$ lies on the boundary of C^0 and C^1 functions spaces for the family $f_\alpha(r, x)$.

It is an easy extension of Proposition 2.2 that for $\alpha > 1$ the scaling laws follow powers of r , i.e. $t \sim r^j$ with $j \in (0, 1)$. We have seen in Table 1 that for $\alpha = 1$ the scaling is logarithmic. Hence the situation observed near $\alpha = 1$, providing a transition from a power law to a constant via a logarithmic “break-point”, resembles the situation of classical qualitative bifurcation theory in the context of a quantitative feature of a dynamical system.

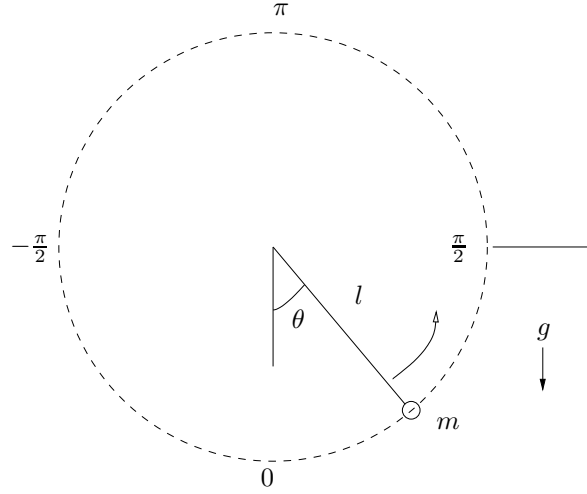


Figure 2: Illustration of the simple pendulum with a barrier at $\frac{\pi}{2}$ at distance 1 from the center.

4 A Model Problem

Consider a pendulum as shown in Figure 2.

We use the following notation: l is the length of the pendulum, m is its mass, g is acceleration due to gravity, ν is a viscous damping coefficient, Ω denotes constant forcing, θ is the angle and I denotes the moment of inertia of the pendulum. Then Newton's law gives the equation of motion as:

$$I\theta'' + mgl \sin \theta = \nu(\omega - \theta')$$

If we consider the overdamped limit of very large damping we can neglect the term $I\theta''$ and consider the classical nonuniform oscillator

$$\theta' = \omega - \frac{mgl}{\nu} \sin \theta \quad (12)$$

For simplicity let us set $L = mgl/\nu$, then (12) reads:

$$\theta' = \omega - L \sin \theta$$

Usually one assumes that the length of the pendulum is fixed and the bifurcation parameter is the applied torque. We generalize this approach and assume that we can vary the length L depending on the angle of the pendulum, i.e. $L = L(\theta)$ with $L : S^1 \rightarrow \mathbb{R}^+$. Note that we regard S^1 as $[-\pi, \pi]$ with endpoints identified as indicated in Figure 2. If we assume that we want to vary the length symmetrically on S^1 there will be breakpoints for $L(\theta)$ at $\theta = \pi, \frac{\pi}{2}, 0, -\frac{\pi}{2}$. Furthermore let us assume that the length of the pendulum is limited at one of the breakpoints, say $L(\pi/2) = 1$ (see Figure 2). Let us consider the family of functions:

$$F_a(\theta) = \begin{cases} -1 + (-2/\pi(\theta + \pi/2))^a & \text{for } \theta \in [-\pi, -\frac{\pi}{2}) \\ -1 + (2/\pi(\theta + \pi/2))^a & \text{for } \theta \in [-\frac{\pi}{2}, 0) \\ 1 - (-2/\pi(\theta - \pi/2))^a & \text{for } \theta \in [0, \frac{\pi}{2}) \\ 1 - (2/\pi(\theta - \pi/2))^a & \text{for } \theta \in [\frac{\pi}{2}, \pi] \end{cases}$$

for $a > 0$. Illustrations for $a = \frac{1}{2}, 1, 2$ are given in Figure 3.

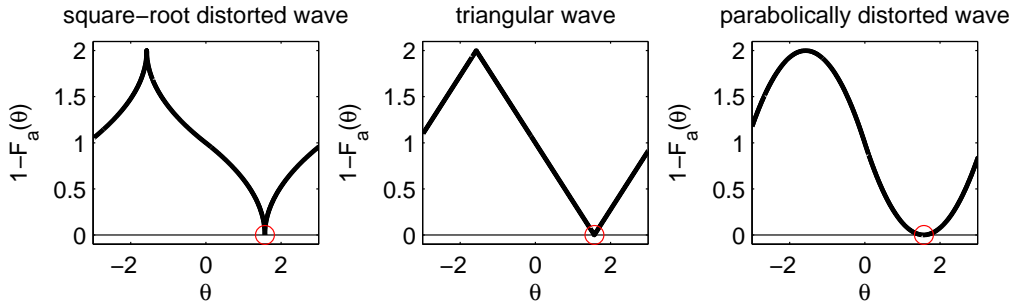


Figure 3: $f(\theta, 1) = 1 + F_a(\theta)$ for $a = \frac{1}{2}, 1, 2$

Now we consider the equation modelling the elongation L as:

$$L(\theta) = \frac{F_a(\theta)}{\sin \theta}$$

Note that this is a priori not defined for $a < 1$ at 0 giving infinite length L . We can simply truncate $L(\theta)$ if we want to construct the experiment but we remark that the form of $L(\theta)$ near 0 is not relevant for the following discussion. It is crucial to notice that $L(\pi/2) = 1$ independent of a , i.e. we pass the bottleneck with the same length. We are left with the equation

$$\dot{\theta} = \omega - F_a(\theta) \quad \text{for } \theta \in [-\pi, \pi] \text{ and } a, \omega \in \mathbb{R}^+ \quad (13)$$

As in the usual sinusoidal case $\dot{\theta} = \omega - \sin \theta$ we have a saddle-node bifurcation at $(\omega, \theta) = (1, \frac{\pi}{2})$ for all $a > 0$ in equation (13) (see also Figure 3). From Theorem 3.2 we see that there exists a quantitative bifurcation for $a = 1$. In particular, $a \in (0, 1)$ gives scaling laws $t \sim O(1)$ as $\omega \rightarrow 1^+$. For the triangular wave $a = 1$, we get a logarithmic scaling and for $a > 1$ we obtain power laws $t \sim (\omega - 1)^j$ for $j \in (0, 1)$.

This means that upon tuning the parameter a in the family F_a and setting ω small and positive we can switch the behaviour of the bottleneck. Hence equation (13) describes how different strategies of varying L affect the motion. The case $0 < a < 1$ corresponds to reducing a long pendulum to unit length for $\theta \rightarrow (\frac{\pi}{2})^-$, the case $a = 1$ corresponds to a very small variation upon approaching $\frac{\pi}{2}$ and $a > 1$ means that we start with a very short pendulum with a rapid increase near $\pi/2$ to reach unit length.

5 Conclusions

We have investigated scaling laws for saddle-node bifurcation in 1-dimensional dynamics. The generic smooth 1-parameter vector field exhibits a square-root scaling law. Dropping the assumptions of genericity we have seen that the nonlinearities of the parameter or the phase-space variable exhibit various power laws. Furthermore we have investigated the case of C^0 vector fields and demonstrated that parameter linearities lead to arbitrary scaling laws. For phase space nonlinearities we found a quantitative bifurcation for a 2-parameter family of C^0 vector fields.

In particular, the theory presented for C^0 vector fields can clearly be extended to more than 1 dimension, but in contrast to the smooth cases we do not have tools like center manifolds immediately

available. As an example it is easy to construct an n -dimensional system, which has locally at least two directions near a saddle-node with different C^0 vector fields in each direction. This substantially complicates the analysis for the natural extension to more dimensions.

We have also demonstrated that a very simple pendulum equation can exhibit a quantitative bifurcation. Many other applications of saddle-node bifurcations occurring in n -dimensional C^0 systems could clearly exhibit the phenomenon.

Furthermore we mention that the different scaling laws found for C^0 vector fields can be directly related to the same scaling laws found in C^0 maps. C^0 maps are of particular relevance in the analysis of discontinuity-induced bifurcations and their associated return maps (Poincaré discontinuity map [PDM] and zero time discontinuity map [ZDM]) [2]. These return maps have different types of singularities, among them we can find piecewise-linear maps and maps with square-root singularities. Hence the methods we presented in this paper are likely to be very useful in the analysis of scaling laws for discontinuity-induced bifurcations.

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